FIBRATIONS AND NULLIFICATIONS

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ABSTRACT

The paper considers some aspects of the preservation of a fibration $F \to E \to B$ by a localization functor L, meaning that $LF \to LE \to LB$ is also a fibration. We obtain necessary and sufficient conditions for this, in case L is equivalent to a nullification functor P_W. Applications include the fact that if such a fibration is preserved by a nullification, then so too are all of its pullbacks. Another application is a homological criterion for a fibration to be preserved. No similar results are known for general localization functors.

0. Introduction

In order to understand the effect of a localization $X \mapsto LX$ on homotopy groups, it is important to know the result of applying the functor L to each of the spaces of a fibration sequence $F \to E \to B$. The key question is whether the resulting sequence $LF \to LE \to LB$ is also a fibration sequence. This was investigated for example in [9].

We recall some standard terminology relating to localization. A **localization** functor L, from the category of spaces of the homotopy type of a CW-complex

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to itself, comes equipped with a natural transformation l: Id $\to L$ that is a coaugmentation, so that for each X we have $Ll_X \simeq l_{LX}: LX \xrightarrow{\simeq} LLX$. Also, one calls X **local** if the coaugmentation map $X \to LX$ is a homotopy equivalence, and a map $g: X \to Y$ is an L-**equivalence** if $Lg: LX \to LY$ is a homotopy equivalence.

Here we 'restrict' attention to localizations of the form $L = L_f$ with respect to some map $f: U \to V$. Such a localization L_f may be defined by specifying that X is local if composition with f induces a weak homotopy equivalence of function spaces $\mathrm{map}(V, X) \to \mathrm{map}(U, X)$. According to [10], whether this apparent restriction is a genuine one depends on the model of set theory being followed; in particular, if Vopěnka's principle is adopted, then every localization may be taken to be of this form.

We say that a fibration sequence $F \to E \xrightarrow{p} B$ (of pointed spaces and maps) **is preserved by** L if the natural map gives a homotopy equivalence $\mathrm{Fib}(Lp) \simeq LF$ between the fibre $\mathrm{Fib}(Lp)$ over the basepoint in LB (being the image under l_B of the basepoint of B) and the localization of the fibre LF. In other words, if applying L gives a commuting diagram of fibration sequences:

$$\begin{tikzcd} F \arrow[r, "l_F"] \arrow[d] & LF \arrow[d] \\ E \arrow[r, "l_E"] \arrow[d, "p"] & LE \arrow[d, "Lp"] \\ B \arrow[r, "l_B"] & LB. \end{tikzcd}$$

In our main result we prove a reduction of preservation of a fibration over B, by a nullification functor $L = P_W$, to preservation of the induced fibration over a 'subspace' $A_L B$ that contains the 'non-local' part of B. We also point out certain conditions for L to be a nullification functor. Recall that for any connected space W the W-**nullification** of X comprises a space $P_W X$ and map $X \to P_W X$ that is initial among maps from X to P_W-local, or W-**null**, spaces Y, namely those Y for which the space of maps $\mathrm{map}(W, Y)$ is weakly homotopy equivalent to Y, or equally, in the pointed category, the pointed function space $\mathrm{map}_*(W, Y)$ is contractible. This corresponds to $L = L_f$ for $f: W \to \mathrm{pt}$. The condition therefore applies in particular not only to $V(n)$-localization, a version of homological localization with respect to Morava K-theories (see, for instance, [9]), but also to the Quillen plus-construction — see [7]. The arguments below generalize those given in [4], [6] for the case where L is the plus-construction.

A version of the main result reads as follows. Let $A_L X$ be the homotopy fibre of the localization's coaugmentation map $l_X: X \to LX$, with the induced map $d_X: A_L X \to X$.

THEOREM 0.1: *Let $F \to E \xrightarrow{p} B$ be a fibration sequence, and let $L = L_f = P_W$ be a homotopy nullification functor. Then the following statements are equivalent.*

(i) *The fibration sequence p is preserved by L.*
(ii) *The pullback $F \to E_1 \xrightarrow{q} A_L B$ of the given fibration sequence p along $d_B: A_L B \to B$ is preserved by L.*
(iii) *The fibrewise localization of q in* (ii) *is a fibre homotopically trivial (that is, a product) fibration sequence.*

The proof is given following the proof of Theorem 4.1 below. Thus the theorem identifies an induced fibration sequence over $A_L B$ as the sole and full obstruction to the preservation of the given fibration sequence by L.

A key property of nullification follows: If P_W preserves a fibration sequence, then it preserves all its pullbacks.

THEOREM 0.2: *Let $F \to E \xrightarrow{p} B$ be a fibration sequence, and let $L = L_f = P_W$ be a homotopy nullification functor. If this sequence is preserved by L, then any fibration sequence induced from p via a map $X \to B$ is also preserved by L.*

We are able to give two proofs of this result: one direct, the other a consequence of our main theorem (4.1) below.

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1. Preliminaries

All spaces are assumed to have the homotopy type of a pointed CW-complex. Unless explicitly said otherwise we assume that spaces and maps are pointed. It is clear that some of the results hold more generally: for example, in this section one need assume only that base spaces are locally path-connected, in order to know that path-components coincide with components. Later, in order to apply Dold's theorem [12], it suffices to deal with D-spaces [3].

We begin with a discussion of localizations and connectedness. Recall from the Introduction that we have assumed that $L = L_f$ for some map f. In fact, it follows from the definition that if f fails to induce a bijection on the set of components, then every local space is contractible. In that event our results hold for trivial reasons. *We assume that f is a map of connected spaces.* We record a useful effect of this, which shows that L restricts to a localization on connected spaces too.

LEMMA 1.1 ([14] (1.A.e.11)): *If X is a connected space, then so is LX.*

Because fibres (for example, loop spaces) play an important role in this work, we are also obliged to consider localization of non-connected spaces. The following observations are helpful.

LEMMA 1.2: *The coaugmentation map $l_X\colon X \to LX$ induces a bijection on the sets of components of X and LX.*

Proof: Since f is a map of connected spaces, the space S^0 is local. Then the result is a consequence of the bijection between $[X, S^0]$ and $[LX, S^0]$. ∎

LEMMA 1.3: *A space X is local if and only if each component is local.*

Proof: Because it induces a bijection on components, the map $X \to LX$ is a homotopy equivalence if and only if its restriction to each component is. ∎

Another useful tweaking of our definition of localization is to use the associated mapping cylinder cofibration construction to deform the coaugmentation to a cofibration (cf. [5] p. 44, [14] p. 6). (The universal nature of this pushout construction ensures that the functoriality of L is retained.) This enables maps from X to a local space to factor through the coaugmentation $X \to LX$ (the factorization being unique up to homotopy under X).

Turning now to fibrations, we recall that associated to any map $p\colon E \to B$ and basepoint b of B there is the associated Hurewicz fibration, or mapping path fibration, $\tilde{p}\colon P_p \to B$ with E and P_p homotopy equivalent over B. The fibre $\mathrm{Fib}(p) = (\tilde{p})^{-1}(b)$ is known as the **homotopy fibre** of p over b, obtainable also as the pullback

$$\begin{tikzcd} \mathrm{Fib}(p) \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ PB \arrow[r] & B \end{tikzcd}$$

where PB denotes the space of paths in B that originate at b. Here and elsewhere we use the basepoint in B as a (usually implicit) part of the apparatus. It is tempting to concentrate on the case where B is connected; however, because this does not always occur, we make it an explicit extra assumption when we use it. A **fibration sequence**

$$F \xrightarrow{h} \mathrm{Fib}(p) \xrightarrow{\iota} E \xrightarrow{p} (B, b),$$

normally written simply as $F \xrightarrow{\iota h} E \xrightarrow{p} B$, consists of a sequence of maps where $\mathrm{Fib}(p)$ is formed with respect to the basepoint b of B and h is a homotopy equivalence.

An important tool for us is fibrewise localization. The following result is a special case of a general construction that allows for the fibrewise application of any functor that preserves weak homotopy equivalences.

Lemma 1.4 (Fibrewise localization): *If $F \to E \xrightarrow{p} B$ is a fibration sequence, then there is a fibration sequence $LF \to \bar{E} \xrightarrow{\bar{p}} B$ and a commuting diagram of fibration sequences*

$$\begin{tikzcd} F \arrow[r, "l_F"] \arrow[d] & LF \arrow[d] \\ E \arrow[r, "a"] \arrow[d, "p"] & \bar{E} \arrow[d, "\bar{p}"] \\ B \arrow[r, "\mathrm{id}"] & B \end{tikzcd}$$

in which a is an L-equivalence.

Proof: In the case where $F = \mathrm{Fib}(p)$, the result is proved in [14](1.F.3), (1.F.4). The general case follows by naturality. ∎

Lemma 1.5: *If, in a fibration sequence $F \to E \xrightarrow{p} B$, both E and B are local, then so is F.*

Proof: By restricting attention to its basepoint component, we may assume that B is connected. Then when E is also connected, this result is just [14](1.A.8.e.3). Passage to the general case is obtained by twofold application of Lemma 1.3, by consideration of p restricted to each component of E. ∎

The key question for this study is when the homotopy equivalence $F \to \mathrm{Fib}(p)$ gives rise to a homotopy equivalence $LF \to \mathrm{Fib}(Lp)$ (or equally, when $L\,\mathrm{Fib}(p) \to \mathrm{Fib}(Lp)$ is also a homotopy equivalence). In that event we say that the localization L **preserves** the fibration sequence $F \to E \xrightarrow{p} B$.

Examples 1.6: Here are some examples of fibrations not preserved by a certain nullification. Most examples in the literature appear to be of path fibrations $\Omega B \to PB \to B$, where the space B has contractible localization whereas ΩB does not, the localization being with respect to some homology theory. Perhaps the oldest example using extraordinary homology theory is $B = K(G, 2)$ where the homology theory is K-theory [1]. Recently Langsetmo and Stanley have given examples with B of arbitrarily high connectivity, with respect to mod p K-homology [17]. A different kind of example appears in [5] p. 56, where it is shown that for any nonzero ring R (having additive group of all finite matrices MR and direct limit general linear group $\mathrm{GL}\, R$) the fibration

$$B(MR) \to B\,\mathrm{GL}\begin{pmatrix} R & R \\ 0 & R \end{pmatrix} \to B\,\mathrm{GL}\, R \times B\,\mathrm{GL}\, R$$

is not preserved by the plus-construction.

LEMMA 1.7: *In the diagram above of fibrewise localization, L preserves p if and only if L preserves $\bar{p}$.*

Proof: Since a is an L-equivalence, the spaces LE and $L\bar{E}$ are homotopy equivalent. ∎

This result has a quick application that we use later.

LEMMA 1.8: *Any fibration sequence $F \to E \xrightarrow{p} B$ with LF contractible is preserved by L; so if in addition B is connected, then Lp is a homotopy equivalence.*

Proof: In this case $\bar{p}$ is a homotopy equivalence, so that $L\bar{p}$ is too (cf. (1.H.1) of [14]). ∎

However, in order to make further progress, we need to impose a condition on the localization L.

2. When is a localization a nullification?

In this section we discuss a condition that allows one to prove the results of the Introduction concerning preservation of fibration sequences.

This enables us to establish the following recognition principle for localizations of the desired form. First, let $A_L X$ be the homotopy fibre of $X \to LX$ over its given basepoint. In the list below, (i) follows the convention that two localizations are equivalent if they have the same local spaces. Notice that in the proof, in order to deduce (i) from the other conditions, we do need to assume that $L = L_f$ for some map f of connected spaces.

THEOREM 2.1: *For any localization $L = L_f$ with f a map of connected spaces, the following are equivalent.*

(i) *L is equivalent to a nullification.*

(ii) *If, in any fibration sequence $F \to E \xrightarrow{p} B$, both F and B are local, then so is E.*

(iii) *Every fibration sequence $F \to E \xrightarrow{p} B$ with B local is preserved by L.*

(iv) *For every space X the space $LA_L X$ is contractible.*

(v) *For every space X the space $A_L X$ is connected, and $A_L(A_L X) = A_L X$.*

Proof: In the proof we may replace B, E by the component of the basepoint and thus take them to be connected.

(i) $\Rightarrow$ (ii). This is shown in [9](4.2) and [14](1.A.8.e.6).

(ii) $\Rightarrow$ (iii). By Lemma 1.7 it suffices to show that the fibration sequence $LF \to \bar{E} \xrightarrow{\bar{p}} B$ obtained by fibrewise localization is preserved by L. Now (ii) yields that $\bar{E}$ is local. So the fibrewise localization consists entirely of local spaces and is therefore homotopically unchanged by L.

(iii) $\Rightarrow$ (iv). If we apply L to the fibration sequence $A_L X \to X \to LX$ with local base we still get a fibration sequence, by (iii). But L is idempotent, so the fibre $LA_L X$ of $LX \to LLX$ is contractible.

(iv) $\Rightarrow$ (ii). Suppose that $F \to E \xrightarrow{p} B$ is a fibration sequence with E and B connected, and both F and B local. Then since B is local, p must factor through l_E: $E \to LE$ via q: $LE \to B$. This gives rise to the diagram

$$\begin{tikzcd}
F \arrow[rrrr, "m"] \arrow[dr] & & & & \mathrm{Fib}\,(q) \arrow[dl] \\
& E \arrow[rr, "l_E"] \arrow[dr, "p"'] & & LE \arrow[dl, "q"] & \\
& & B & &
\end{tikzcd}$$

By Lemma 1.5, $\mathrm{Fib}(q)$ and then $\mathrm{Fib}(m)$ are both local spaces. However, in this situation (see [11] Lemma 2.1 or [5] p. 35) $\mathrm{Fib}(m)$ coincides with $\mathrm{Fib}(l_E) = A_L E$. Since it is already local, we have $A_L E = L(A_L E)$, which is contractible by assumption. Therefore l_E is a homotopy equivalence, making E local, as required.

(iv) $\Rightarrow$ (v). We again use the fact that the space S^0 is local. Since by universality every map from $A_L X$ to S^0 must factor through the contractible space $LA_L X$, it follows that $A_L X$ is connected. We are therefore able to iterate the construction. Then by (iv), $A_L(A_L X)$ is the homotopy fibre of the localization map from $A_L X$ to the contractible space $LA_L X$. Hence the result.

(v) $\Rightarrow$ (iv). Again we exploit the fact that $A_L(A_L X)$ is the homotopy fibre of the localization map (of connected spaces) $A_L X \to L A_L X$.
(iv) $\Rightarrow$ (i). With $L = L_f$ where $f\colon U \to V$, we let $W = (A_L U) \vee (A_L V)$. We invoke the result [14](1.D.5)(iv) that for any map g

$$L_g(X \vee Y) \simeq L_g(L_g X \vee L_g Y).$$

First, in the case $g = f$ it immediately shows, using (iv), that $LW \simeq \mathrm{pt}$. Then by [14](1.C.5) every L_f-local space is W-null. Second, with $g\colon W \to \mathrm{pt}$ it yields that

$$P_W(P_W(A_L U) \vee P_W(A_L V)) \simeq P_W W \simeq \mathrm{pt},$$

the latter equivalence following from the definition. So we deduce from [14] (1.A.8.e.9) that

$$\mathrm{map}_*(P_W(A_L U) \vee P_W(A_L V), P_W(A_L U)) \simeq \mathrm{pt},$$

which implies that $P_W(A_L U) \simeq \mathrm{pt}$, and similarly for $P_W(A_L V)$. Thus in the commuting square

$$\begin{tikzcd} P_W U \arrow[r, "P_W f"] \arrow[d] & P_W V \arrow[d] \\ P_W LU \arrow[r, "P_W Lf"] & P_W LV \end{tikzcd}$$

both vertical arrows are homotopy equivalences. After [14](1.C.5), Lf is already a homotopy equivalence. This leaves $P_W f$ as a homotopy equivalence, and by [14](1.C.5) again every W-null space is L_f-local. ∎

Here is a class of localizations for which the above conditions are satisfied.

COROLLARY 2.2: *Let $\mathcal{X}$ be a class of groups (called $\mathcal{X}$-groups) closed under passage to*

(i) *group extensions of $\mathcal{X}$-groups by $\mathcal{X}$-groups,*
(ii) *subgroups of $\mathcal{X}$-groups, and*
(iii) *quotients of $\mathcal{X}$-groups by central subgroups.*

If L is a localization having as local spaces precisely the spaces all of whose homotopy groups lie in $\mathcal{X}$, then L satisfies Theorem 2.1(ii).

Proof: The proof follows directly from the group extensions (with $i \geq 1$)

$$\mathrm{Im}[\pi_i(F) \to \pi_i(E)] \rightarrowtail \pi_i(E) \twoheadrightarrow \mathrm{Ker}[\pi_i(B) \to \pi_{i-1}(F)]$$

that arise in the exact homotopy sequence of a fibration sequence $F \to E \to B$. (Recall that the extension

$$\mathrm{Im}[\pi_{i+1}(B) \to \pi_i(F)] \rightarrowtail \pi_i(F) \twoheadrightarrow \mathrm{Im}[\pi_i(F) \to \pi_i(E)]$$

is always central.) ∎

In particular, this corollary affords an immediate proof that the plus-construction satisfies Theorem 2.1(ii), for its local spaces are just those whose homotopy groups lack nontrivial perfect subgroups. A less direct proof can be obtained by expressing the plus-construction in the form P_W for suitable connected W [14], [7].

Example 2.3: Suppose that the localization $L = L_f$ has underlying map $f\colon S^2 \to S^2$ of degree p. It is shown in [14](1.H.4) that $LA_L(S^2)$ is not simply-connected. So by the theorem, L cannot be equivalent to a nullification.

3. Localizations and homotopy pullbacks

Henceforth we assume that the localization $L = P_W$ satisfies the conditions of Theorem 2.1 above.

PROPOSITION 3.1: *If a pointed fibration sequence $F \to E \xrightarrow{p} B$, with F connected, is preserved by L, then any pointed fibration sequence p_1 induced from p via a map $B_1 \to B$ is also preserved by L.*

Proof: We can assume that B_1 is connected — otherwise we work with the basepoint component of B_1. As in [11] Lemma 2.1, there is a diagram

$$\begin{tikzcd}
A_LF \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & LF \arrow[d] \\
GTf_1 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "p_1"] & LE \arrow[d, "Lp"] \\
H \arrow[r] & B_1 \arrow[r] & LB
\end{tikzcd}$$

that depicts the induced fibration sequence p_1 and its map, via composition, to the localized fibration sequence Lp. The rest of the diagram is determined by taking homotopy fibres of the horizontal right-hand maps, and by [11] has the property that A_LF is also the homotopy fibre of the induced map f_1. Since (by 2.1(iv)) $L(A_LF)$ is contractible, Lemma 1.8 shows that the fibration sequence f_1

is preserved by L. Also, we know from Theorem 2.1(iii) that all the horizontal fibration sequences are preserved by L, since they have local bases.

Then localization of the lower right square gives rise, as in [11] again, to the fibration sequence diagram below.

$$\begin{tikzcd}
\mathrm{Fib}(Lf_1) \arrow[r] \arrow[d] & \mathrm{Fib}(Lp_1) \arrow[r] \arrow[d] & LF \arrow[d] \\
LG \arrow[r] \arrow[d, "Lf_1"] & LE_1 \arrow[r] \arrow[d, "Lp_1"] & LE \arrow[d, "Lp"] \\
LH \arrow[r] & LB_1 \arrow[r] & LB
\end{tikzcd}$$

This reveals that the map from $\mathrm{Fib}(Lp_1)$ to LF has contractible fibre $\mathrm{Fib}(Lf_1) \simeq L(A_L F)$, and so, since LF is connected by Lemma 1.1, is a homotopy equivalence. Therefore our induced fibration sequence p_1 is preserved by L, as claimed. ∎

We now present a partial converse to the above result. It will be crucial to our main argument.

LEMMA 3.2: *Suppose that*

$$\begin{tikzcd}
D \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"'] & \\
A \arrow[r] & B \arrow[r, "r"] & C
\end{tikzcd}$$

comprises a homotopy-Cartesian square along with a fibration sequence $A \to B \xrightarrow{r} C$ where C is local. Then:

(a) *There is a homotopy equivalence of homotopy fibres:* $\mathrm{Fib}(Lq) \simeq \mathrm{Fib}(Lp)$;

(b) *p is preserved by L if and only if q is preserved by L.*

Proof: For the diagram with both squares homotopy-Cartesian

$$\begin{tikzcd}
D \arrow[r, "q"] \arrow[d] & A \arrow[r] \arrow[d] & PC \arrow[d] \\
E \arrow[r, "p"] & B \arrow[r, "r"] & C
\end{tikzcd}$$

we have from Theorem 2.1(iii) that the mapping path fibrations of both r and

$r \circ p$ are preserved by L. So there is also a commuting diagram

$$\begin{tikzcd} LD \arrow[r] \arrow[d, "Lq"] & LE \arrow[d, "Lp"] \\ LA \arrow[r] \arrow[d] & LB \arrow[d, "Lr"] \\ PLC \arrow[r] & LC \end{tikzcd}$$

with the outer rectangle and lower square homotopy-Cartesian. These combine to give the upper square homotopy-Cartesian as well. This gives (a). Then we have $\mathrm{Fib}(Lq) \simeq \mathrm{Fib}(Lp)$ and $\mathrm{Fib}(q) \simeq \mathrm{Fib}(p)$. So $\mathrm{Fib}(Lp) \simeq L\,\mathrm{Fib}(p)$ precisely when $\mathrm{Fib}(Lq) \simeq L\,\mathrm{Fib}(q)$, which is the content of (b). ∎

4. Main results

We now use part of the above to prove the main result of the Introduction. First, a reminder and some notation.

Recall [15] that, with $\operatorname{aut} F$ (respectively $\operatorname{aut}^\circ F$) as the monoid of self-equivalences (respectively pointed self-equivalences) of a connected space F, there is a universal fibration

$$F \longrightarrow B\operatorname{aut}^\circ F \longrightarrow B\operatorname{aut} F.$$

If a basepoint has not already been specified, we may choose one so as to make pointed the space F, and thereby any fibration sequence commencing with F. We also exploit the easy fact, proved via Lemma 1.2, that if a fibration sequence is preserved by L, then so is any other fibration sequence fibre homotopy equivalent to it. Thus a fibration sequence $F \to E \xrightarrow{p} B$ gives rise to a pullback of fibration sequences

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}"] \arrow[d] & F \arrow[d] \\ E \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner"] & B\operatorname{aut}^\circ F \arrow[d] \\ B \arrow[r, "\hat{p}"] & B\operatorname{aut} F \end{tikzcd}$$

while functoriality gives a commuting diagram [14](1.F.1.2)

$$\begin{tikzcd} F \arrow[r, "l_F"] \arrow[d] & LF \arrow[d] \\ B\,\mathrm{aut}^\circ F \arrow[r] \arrow[d] & B\,\mathrm{aut}^\circ LF \arrow[d] \\ B\,\mathrm{aut}\,F \arrow[r, "\hat{\zeta}"] & B\,\mathrm{aut}\,LF \end{tikzcd}$$

Moreover [14](1.F.1.4), we have $\hat{\zeta} \circ \hat{p} = (\bar{p})\hat{}$, or equally, the fibration sequence $LF \to \bar{E} \xrightarrow{\bar{p}} B$ is induced from the fibration sequence

$$LF \to \overline{B\,\mathrm{aut}^\circ F} \xrightarrow{\zeta} B\,\mathrm{aut}\,F,$$

where ζ is obtained by pulling back over $\hat{\zeta}$ the universal LF-fibration sequence. More generally, the above considerations apply to non-connected F as well, provided that one makes the appropriate replacement for the universal fibration sequence over $B\,\mathrm{aut}\,F$ [2].

THEOREM 4.1: *Let $F \to E \xrightarrow{p} B$ be a fibration sequence. Then the following statements are equivalent.*

(i) *The fibration sequence p is preserved by L.*

(ii) *The composite*

$$A_L B \xrightarrow{d_B} B \xrightarrow{\hat{p}} B\,\mathrm{aut}\,F \xrightarrow{\hat{\zeta}} B\,\mathrm{aut}\,LF$$

is nullhomotopic.

(iii) *The pullback $LF \to E_1 \xrightarrow{q} A_L B$ of the fibration sequence*

$$LF \to \bar{E} \xrightarrow{\bar{p}} B$$

over d_B is fibre homotopy trivial and thus $LF \simeq LE_1$.

Proof: $\hat{\zeta} \circ \hat{p} \circ d_B$ classifies the pullback $LF \to E_1 \xrightarrow{q} A_L B$ of the fibration sequence $LF \to \bar{E} \xrightarrow{\bar{p}} B$ over d_B. So (ii) and (iii) are equivalent. We therefore have to show that this fibration sequence is fibre homotopy trivial precisely when p is preserved by L, or equivalently, by Lemma 1.7, when $\bar{p}$ is preserved by L. However, by Lemma 3.2 applied to the diagram

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "q"] & \bar{E} \arrow[d, "\bar{p}"] & \\ A_L B \arrow[r, "d_B"] & B \arrow[r, "l_B"] & LB \end{tikzcd}$$

this last is equivalent to its pullback q over d_B being preserved by L. Now by Theorem 2.1(iv), $L(A_L B)$ is contractible. So q is preserved by L just when there is a map of fibration sequences:

$$\begin{tikzcd} LF \arrow[r] \arrow[d, "="'] & E_1 \arrow[r] \arrow[d] & A_L B \arrow[d] \\ LF \arrow[r, "\simeq"] & LE_1 \arrow[r] & \mathrm{pt} \end{tikzcd}$$

Since by the Dold theorem [12] any fibration sequence is fibre homotopy trivial if and only if the inclusion of the fibre admits a left homotopy inverse (cf. [16](4.3)), it follows that q is preserved by L precisely when it is a fibre homotopy trivial fibration sequence. ∎

Proof of Theorem 0.1: The equivalence of (i) and (ii) is immediate from Lemma 3.2. The equivalence of (ii) and (iii) follows by applying the equivalence of Theorem 4.1(i), (iii) to the fibration sequence q instead of p, using the fact (2.1(v)) that $A_L(A_L B) = A_L B$. ∎

Proof of Theorem 0.2: First, note that if $B \xrightarrow{\hat{p}} B\,\mathrm{aut}\,F$ is the classifying map for a fibration sequence over B, then the composite $X \to B \xrightarrow{\hat{p}} B\,\mathrm{aut}\,F$ is the classifying map for the induced fibration sequence over X. Thus, from the equivalence of (i) and (ii) in (4.1), we are reduced to showing that if the composite

$$A_L B \xrightarrow{d_B} B \xrightarrow{\hat{p}} B\,\mathrm{aut}\,F \xrightarrow{\hat{\zeta}} B\,\mathrm{aut}\,LF$$

is nullhomotopic, then so is the composite

$$A_L X \xrightarrow{d_X} X \to B \xrightarrow{\hat{p}} B\,\mathrm{aut}\,F \xrightarrow{\hat{\zeta}} B\,\mathrm{aut}\,LF.$$

However, naturality of the coaugmentation implies that there is a commuting square

$$\begin{tikzcd} A_L X \arrow[r] \arrow[d, "d_X"] & A_L B \arrow[d, "d_B"] \\ X \arrow[r] & B, \end{tikzcd}$$

whence the result follows. ∎

5. Application

To conclude, we provide an example of application of our main results. We begin with a lemma that provides a link between acyclic and nilpotent spaces.

LEMMA 5.1: *The following conditions on a connected space A are equivalent.*

(i) *A is acyclic.*

(ii) *There are no essential maps from A to any space whose fundamental group contains no nontrivial perfect subgroup.*

(iii) *There are no essential maps from A to any nilpotent space nor to any homotopy limit of a diagram of nilpotent spaces.*

(iv) *There are no essential maps from A to any Eilenberg–MacLane space of the form $K(\mathbb{Q}, n)$ or $K(C_p, n)$, where $n \geq 1$ and C_p denotes a cyclic group of prime order.*

Moreover, these conditions imply that no essential map from A induces the trivial map on fundamental groups.

Proof: To argue from (i) to (ii), apply the plus-construction to a map from A to a space Y. In the commuting diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "q_A"] & Y \arrow[d, "q_Y"] \\ A^+ \arrow[r] & Y^+ \end{tikzcd}$$

if A is acyclic then A^+ is contractible, while if the fundamental group of Y lacks nontrivial perfect subgroups, then $Y^+ = Y$. Reduction from (ii) to (iii), and from (iii) to (iv), is just an obvious specialization.

The deduction of (i) from (iv) is of course via universal coefficient arguments; however, one has to take care to avoid assumptions that homology has finite type. Proceeding as in [8], we first use triviality of each $[A, K(\mathbb{Q}, n)]$ to note the vanishing of rational cohomology in all positive dimensions. It follows by universal coefficients that all the groups $\mathrm{Hom}(H_n(A;\mathbb{Z}), \mathbb{Q})$ are trivial, and thus that the groups $H_n(A;\mathbb{Z})$ are torsion. On the other hand, triviality of each $H^n(A; C_p) = [A, K(C_p, n)]$ ensures that of its direct summands

$$\mathrm{Hom}(H_n(A;\mathbb{Z}), C_p) \quad \text{and} \quad \mathrm{Ext}(\tilde{H}_{n-1}(A;\mathbb{Z}), C_p).$$

The vanishing of the former implies that multiplication by p on $H = H_n(A;\mathbb{Z})$ is always surjective. Therefore there is an exact sequence

$$0 \to {}_pH \to H \xrightarrow{p} H \to 0;$$

its Hom-Ext sequence gives $\mathrm{Hom}({}_pH, C_p) = 0$, so that ${}_pH = 0$. Hence multiplication by p on H is an isomorphism. This being true for all primes p, and H being a torsion group, H must be trivial. Hence A is acyclic.

For the final assertion, we use the fact that any map $A \to Y$ that is trivial on fundamental groups must factor through the plus-construction space A^+. However, acyclicity of A makes A^+ contractible. ∎

The following generalizes Theorem 1.1(b) of [4]. However, since the proof strategy is different here, even in that case it affords a new proof. In essence the result deals with fibration sequences over acyclic spaces, which in general will not be preserved by localizations — think of the path fibration. However, in the following special example preservation is guaranteed.

Corollary 5.2: *Let $F \to E \xrightarrow{p} B$ be a fibration sequence of connected spaces, with the following properties.*

(i) *LF is a nilpotent space.*
(ii) *$\mathrm{Im}(\pi_1(A_LB) \to \pi_1(B))$ acts nilpotently on $H_*(F;\mathbb{Z})$.*
(iii) *A_LB is acyclic.*

Then p is preserved by L.

Proof: The action of $\pi_1(A_LB)$ on $H_*(F;\mathbb{Z})$ is induced from that of $\pi_1(B)$. Then the action of $\pi_1(A_LB)$ on $H_*(LF;\mathbb{Z})$ is induced from that on $H_*(F;\mathbb{Z})$, and hence by (ii) is also nilpotent. So, because of (i), we can apply the last sentence of [15], at least for each Postnikov section P_nLF of LF. For then the composite

$$A_LB \xrightarrow{d_B} B \xrightarrow{\hat{p}} B\,\mathrm{aut}\,F \xrightarrow{\hat{\zeta}} B\,\mathrm{aut}\,LF \to B\,\mathrm{aut}(P_nLF)$$

factors through a nilpotent space, and so by Lemma 5.1 must be nullhomotopic. To see that the map to $B\,\mathrm{aut}\,LF$ must also be nullhomotopic, we apply the short exact sequence of [13]:

$$\mathrm{proj\,lim}^1\,\pi_2(B\,\mathrm{aut}(P_nLF)) \rightarrowtail \pi_1(B\,\mathrm{aut}\,LF) \twoheadrightarrow \mathrm{proj\,lim}\,\pi_1(B\,\mathrm{aut}(P_nLF)).$$

Here the induced homomorphism on fundamental groups must map $\pi_1(A_LB)$ into the abelian kernel of the sequence. Since $\pi_1(A_LB)$ is however a perfect group, the homomorphism must be trivial. So, by the lemma again, the map from A_LB to $B\,\mathrm{aut}\,LF$ is after all nullhomotopic too. The result is then immediate from Theorem 4.1. ∎

It would be interesting to see whether this result has a counterpart in other homology theories.

References

[1] D. W. Anderson and L. Hodgkin, *The K-theory of Eilenberg–MacLane complexes,* Topology **7** (1968), 317–329.

[2] M. G. Barratt, V. K. A. M. Gugenheim and J. C. Moore, *On semisimplicial fibre-bundles,* American Journal of Mathematics **81** (1959), 639–657.

[3] A. J. Berrick, *The Samelson ex-product*, The Quarterly Journal of Mathematics. Oxford Series 2 **27** (1976), 173–180.

[4] A. J. Berrick, *The plus-construction and fibrations,* The Quarterly Journal of Mathematics. Oxford Series 2 **33** (1982), 149–157.

[5] A. J. Berrick, *An Approach to Algebraic K-Theory,* Research Notes in Mathematics 56, Pitman, London, 1982.

[6] A. J. Berrick, *Characterization of plus-constructive fibrations,* Advances in Mathematics **48** (1983), 172–176.

[7] A. J. Berrick and C. Casacuberta, *A universal space for plus-constructions,* Topology **38** (1999), 467–477.

[8] A. J. Berrick and W. G. Dwyer, *The spaces that define algebraic K-theory,* Topology **39** (2000), 225–228.

[9] A. K. Bousfield, *Localization and periodicity in unstable homotopy theory,* Journal of the American Mathematical Society **7** (1994), 831–873.

[10] C. Casacuberta, D. Scevenels and J. H. Smith, *Implications of large-cardinal principles in homotopical localization,* preprint.

[11] F. R. Cohen, J. C. Moore and J. A. Neisendorfer, *The double suspension and exponents of the homotopy groups of spheres,* Annals of Mathematics **110** (1979), 549–565.

[12] A. Dold, *Partitions of unity in the theory of fibrations,* Annals of Mathematics **78** (1963), 223–255.

[13] E. Dror, W. G. Dwyer and D. M. Kan, *Self-homotopy equivalences of Postnikov conjugates,* Proceedings of the American Mathematical Society **74** (1979), 183–186.

[14] E. Dror Farjoun, *Cellular Spaces, Null Spaces and Homotopical Localization,* Lecture Notes in Mathematics **1622**, Springer, Berlin, 1996.

[15] E. Dror Farjoun and A. Zabrodsky, *Unipotency and nilpotency in homotopy equivalences,* Topology **18** (1979), 187–197.

[16] I. M. James, *The Topology of Stiefel Manifolds,* London Mathematical Society Lecture Notes 24, Cambridge University Press, London, 1976.

[17] L. Langsetmo and D. Stanley, *Non-durable K-theory equivalence and Bousfield localization,* K-Theory **24** (2001), 397–410.